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On the difference between the Lorentz and Ampère force laws in magnetostatics

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Abstract. The purpose of this paper is to review the equivalence, or the difference, of the two possible force laws in magnetostatics for interacting particles: namely the Ampère and Lorentz force laws. We show that these two laws are mathematically different and physically correct since they apply to two different systems: the former must be used for a closed system and the latter for an open system.

1. Introduction

There is an ongoing controversy between the followers of the Ampère force [1-5] and the followers of the Lorentz force [6-14] concerning the equivalence of the two laws in magnetostatics. A current element $I' \, \mathrm{d}\boldsymbol{r}'$ located at the point $\boldsymbol{r}'$ will exert a Lorentz force on another current element $I \, \mathrm{d}\boldsymbol{r}$ located at the point $\boldsymbol{r}$ given by

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{L}} = \frac{II'}{c^2R^2}\,\mathrm{d}\boldsymbol{r} \wedge (\mathrm{d}\boldsymbol{r}' \wedge \boldsymbol{n}) \qquad \text{with } \boldsymbol{n} = \frac{\boldsymbol{r}-\boldsymbol{r}'}{R} = \frac{\boldsymbol{R}}{R}. \tag{1}$$

Conversely the force exerted by the element $I \, \mathrm{d}\boldsymbol{r}$ on the element $I' \, \mathrm{d}\boldsymbol{r}'$ is

$$\mathrm{d}^2\boldsymbol{F}'_{\mathrm{L}} = \frac{II'}{c^2R^2}\,\mathrm{d}\boldsymbol{r}' \wedge (\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{n}') \qquad \text{with } \boldsymbol{n} = -\boldsymbol{n}'. \tag{2}$$

In contrast the law proposed by Ampère has the expression:

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{A}} = \frac{II'}{c^2R^2}[3(\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{n})(\mathrm{d}\boldsymbol{r}'\cdot\boldsymbol{n}) - 2(\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}')]\boldsymbol{n}. \tag{3}$$

Knowing that

$$\mathrm{d}\boldsymbol{r} \wedge (\mathrm{d}\boldsymbol{r}' \wedge \boldsymbol{n}) = (\mathrm{d}\boldsymbol{r}\cdot\boldsymbol{n})\,\mathrm{d}\boldsymbol{r}' - (\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}')\boldsymbol{n} \tag{4}$$

$$\mathrm{d}\boldsymbol{r}' \wedge (\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{n}') = (\mathrm{d}\boldsymbol{r}'\cdot\boldsymbol{n}')\,\mathrm{d}\boldsymbol{r} - (\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}')\boldsymbol{n}' \tag{5}$$

we see that the above formula are not symmetric with respect to the line-current elements $\mathrm{d}\boldsymbol{r}$ and $\mathrm{d}\boldsymbol{r}'$ and therefore the Lorentz force does not obey Newton's third law of the equality of action and reaction: $\mathrm{d}^2\boldsymbol{F}_{\mathrm{L}} = -\mathrm{d}^2\boldsymbol{F}'_{\mathrm{L}}$. On the contrary the square bracket term in formula (3) is symmetric. Therefore the Ampère law verifies Newton's third law of the equality of action and reaction: $\mathrm{d}^2\boldsymbol{F}_{\mathrm{A}} = -\mathrm{d}^2\boldsymbol{F}'_{\mathrm{A}}$ as a consequence of the change of sign of $\boldsymbol{n} = -\boldsymbol{n}'$. For closed current loops C and C', the total Lorentz force is

$$\boldsymbol{F}_{\mathrm{L}} = \frac{II'}{c^2}\int_{C'}\mathrm{d}\boldsymbol{r}'\int_C\frac{\boldsymbol{n}\cdot\mathrm{d}\boldsymbol{r}}{R^2} - \int_{C'}\int_C\frac{(\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}')}{R^2}\boldsymbol{n} \tag{6}$$

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since

$$\int_C \frac{\boldsymbol{n}\cdot \mathrm{d}\boldsymbol{r}}{R^2} = -\int_C \nabla\left(\frac{1}{R}\right)\cdot \mathrm{d}\boldsymbol{r} = 0 \tag{7}$$

the term in the Lorentz force which does not follow Newton's third law cancels for two closed current loops in interaction, provided that the current elements belong to two different circuits. Consequently the Ampère and Lorentz force laws are equivalent in that case.

2. Self-interaction

These formulae can also be used to calculate the interaction of one finite current element with all the others in the same circuit. In that case the other elements will no longer form a closed loop and therefore the two formulations of the force will differ. Now, if one insists on forming a closed loop to calculate the self-interaction of the circuit, then the above integrals will present singularities for $R = 0$ which must be taken into account.

Rather than considering line elements one can compare the two laws in a form involving volume-current distributions. To obtain these laws we start from the potential vector which is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \int_{C'} \frac{I'}{cR}\,\mathrm{d}\boldsymbol{r}'. \tag{8}$$

Knowing that

$$I' = \int_{S'} \boldsymbol{J}'\cdot \mathrm{d}\boldsymbol{S}$$

a priori we have

$$\int_{C'} \frac{\mathrm{d}\boldsymbol{r}'}{cR}\int_{S'} \boldsymbol{J}'\cdot \mathrm{d}\boldsymbol{S} \neq \int_{V'} \frac{\boldsymbol{J}'}{cR}\,\mathrm{d}V'. \tag{9}$$

In magnetostatics, since there is no density current perpendicular to the surface of the conductor, we can assume the equality of the above equation and therefore the Lorentz force becomes

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{L}} = \frac{1}{c^2R^2}\boldsymbol{J}\wedge(\boldsymbol{J}'\wedge\boldsymbol{n})\,\mathrm{d}V\,\mathrm{d}V' \tag{10}$$

which can be rewritten in the form:

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{L}} = \frac{1}{c^2R^2}[(\boldsymbol{J}\cdot\boldsymbol{n})\boldsymbol{J}' - (\boldsymbol{J}\cdot\boldsymbol{J}')\boldsymbol{n}]\,\mathrm{d}V\,\mathrm{d}V'. \tag{11}$$

For the Ampère force we have

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{A}} = \frac{\boldsymbol{n}}{c^2R^2}[3(\boldsymbol{J}\cdot\boldsymbol{n})(\boldsymbol{J}'\cdot\boldsymbol{n}) - 2(\boldsymbol{J}\cdot\boldsymbol{J}')]\,\mathrm{d}V\,\mathrm{d}V'. \tag{12}$$

Now we can calculate the difference between the two laws for the two cases related to the asymmetry of the Lorentz law which are:

$$\Delta \boldsymbol{F} = (\mathrm{d}^2\boldsymbol{F}_{\mathrm{A}} - \mathrm{d}^2\boldsymbol{F}_{\mathrm{L}})/\mathrm{d}V\,\mathrm{d}V'$$

$$\Delta \boldsymbol{F}' = (\mathrm{d}^2\boldsymbol{F}'_{\mathrm{A}} - \mathrm{d}^2\boldsymbol{F}'_{\mathrm{L}})/\mathrm{d}V\,\mathrm{d}V'$$

with

$$\Delta \boldsymbol{F} = \frac{1}{c^2R^2}[3(\boldsymbol{J}\cdot\boldsymbol{n})(\boldsymbol{J}'\cdot\boldsymbol{n})\boldsymbol{n} - (\boldsymbol{J}\cdot\boldsymbol{J}')\boldsymbol{n} - (\boldsymbol{J}\cdot\boldsymbol{n})\boldsymbol{J}'] \tag{13}$$

$$\Delta \boldsymbol{F}' = \frac{1}{c^2R^2}[3(\boldsymbol{J}\cdot\boldsymbol{n})(\boldsymbol{J}'\cdot\boldsymbol{n})\boldsymbol{n}' - (\boldsymbol{J}\cdot\boldsymbol{J}')\boldsymbol{n}' - (\boldsymbol{J}'\cdot\boldsymbol{n}')\boldsymbol{J}]. \tag{14}$$

Let us define the quantities:

$$\boldsymbol{K} = \boldsymbol{J}\cdot\nabla'\left(\frac{1}{R}\right)\boldsymbol{J}' \qquad \boldsymbol{K}' = \boldsymbol{J}'\cdot\nabla\left(\frac{1}{R}\right)\boldsymbol{J}. \tag{15}$$

Knowing that

$$\nabla\cdot\boldsymbol{K}' = \nabla'\cdot\boldsymbol{K} = \frac{1}{R^3}[3(\boldsymbol{J}\cdot\boldsymbol{n})(\boldsymbol{J}'\cdot\boldsymbol{n}) - (\boldsymbol{J}\cdot\boldsymbol{J}')] \tag{16}$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\boldsymbol{n}}{R^2} \qquad \nabla\left(\frac{1}{R}\right) = \frac{\boldsymbol{n}'}{R^2} \tag{17}$$

if we substitute equations (16) and (17) in equations (13) and (14) we obtain

$$\Delta \boldsymbol{F} = \frac{1}{c^2}(\boldsymbol{R}\nabla'\cdot\boldsymbol{K} - \boldsymbol{K}) \tag{18}$$

$$\Delta \boldsymbol{F}' = \frac{1}{c^2}(-\boldsymbol{R}\nabla\cdot\boldsymbol{K}' - \boldsymbol{K}'). \tag{19}$$

Since the quantities $\boldsymbol{K}$ and $\boldsymbol{K}'$ are of opposite sign this implies that the differences $\Delta\boldsymbol{F}$ and $\Delta\boldsymbol{F}'$ cannot be equal due to the presence of the term which is at the origin of the discrepancy between the two force laws.

The difference of the forces exerted on the current volume $\mathrm{d}V'$ by the volume V is obtained by integrating equation (19) over V. This may be done by using the integral identity:

$$\int_V (\boldsymbol{R}\nabla\cdot\boldsymbol{K}' + \boldsymbol{K}')\,\mathrm{d}V = \int_S (\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{K}')\boldsymbol{R} \tag{20}$$

where S is the surface enclosing V. Therefore, using the definition of $\boldsymbol{K}'$, the surface integral becomes

$$\boldsymbol{I} = \int_S (\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{K}')\boldsymbol{R} = -\int_S (\boldsymbol{J}'\cdot\boldsymbol{R})(\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{S})\frac{\boldsymbol{R}}{R^3}. \tag{21}$$

Now we can follow the approach of Christodoulides [9] concerning the calculation of the surface integral, which consists of the outer surface of the current conductor on which we have the condition $\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{S} = 0$ and two surfaces S_1 and S_2 produced by sections of the conductor. For a closed current loop, the sum of the integrals on the surfaces S_1 and S_2 will cancel since the surface elements $\mathrm{d}\boldsymbol{S}$ on S_1 and S_2 have opposite directions, provided that the current volume $\mathrm{d}V'$ is excluded from the volume V.

If we incorporate the volume $\mathrm{d}V'$ inside the closed volume V, we have to take into account the singularity of the integral. We cannot use Christodoulides's approach by enclosing the current distribution $\mathrm{d}V'$ by a sphere since the quantity $\boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S}$ is $-J_r$ on one half of the sphere and J_r on the other half as shown in figures 1 and 2 of [9]. This is the reason why Christodoulides finds zero in his equation (10) because the quantity $\boldsymbol{J}\cdot\boldsymbol{n}$ keeps the same sign on the sphere, which is not the case. Therefore we must adopt cylindrical symmetry in this problem and calculate the surface integral on a cylindrical box which is represented in figure 1 for the thickness of the box tending to zero. This calculation has been done by Bouix [15, p 153] for a scalar field. If we assume that the transverse part of $\boldsymbol{J}'$, which brings the current from the source to the circuit, does not give a contribution to the integral then we can use Bouix's approach for the current directed in the z direction only. It is important to remark that for the box of figure 1 the quantity $(\boldsymbol{J}'\cdot\boldsymbol{R})(\boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S})$ does not change its sign on the surfaces S_1 and S_2.

For cylindrical coordinates, we have the following definitions:

$$OA = a = R\cos\theta \qquad OM = r = a\tan\theta \qquad AM = \boldsymbol{R} = \frac{a}{\cos\theta}\boldsymbol{n} \tag{22}$$

and

$$\mathrm{d}S = \tfrac{1}{2}\mathrm{d}(r^2)\,\mathrm{d}\phi = \tfrac{1}{2}\mathrm{d}(a^2\tan^2\theta)\,\mathrm{d}\phi$$
$$\mathrm{d}S = a^2\frac{\sin\theta}{\cos^3\theta}\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{23}$$

with

$$\boldsymbol{J}\cdot \mathrm{d}\boldsymbol{S} = J\,\mathrm{d}S \qquad \boldsymbol{J}'\cdot\boldsymbol{R} = -J'R\cos\theta.$$

With the above definitions the integral $\boldsymbol{I}$ on a surface S becomes

$$\boldsymbol{I} = \int_S JJ'\frac{\cos\theta}{R}\boldsymbol{n}\,\mathrm{d}S. \tag{24}$$

Knowing that for surfaces S_1 and S_2 the r–z components of the vector $\boldsymbol{n}$ are $\sin\theta$,

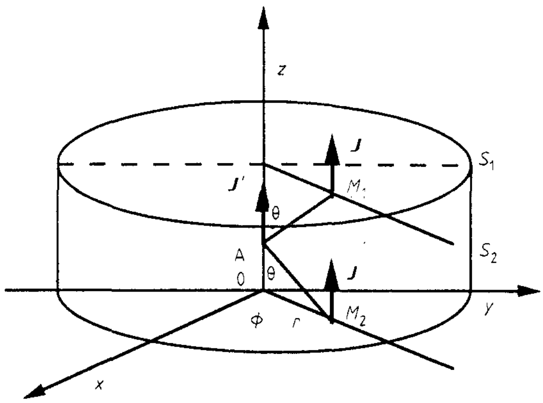

Figure 1. The cylindrical box defining the surface S over which the integral (24) is evaluated.

$\pm\cos\theta$ then the r-z components of the integral $\boldsymbol{I}$ are given by

$$I_r = 2\pi a \int_0^{\theta_0} JJ' \frac{\sin^2\theta}{\cos\theta}\,d\theta = 2\pi aJJ'\left[-\sin\theta_0 + \ln\tan\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\right] \tag{25}$$

$$I_z = \pm 2\pi a \int_0^{\theta_0} JJ' \sin\theta\,d\theta = \pm 2\pi aJJ'(\cos\theta_0 - 1). \tag{26}$$

In the above integrals the angle θ_0 is a function of the distance a which tends to $\pi/2$ for $a \to 0$.

Therefore the quantity $a \ln\tan(\pi/4 + \theta_0/2)$ which is an indeterminate form tends to infinity for $a \to 0$. This can be demonstrated by using l'Hospital's rule. Thus the sum of the integrals I_z on the surfaces S_1 and S_2 is zero whatever the value of a. In contrast the sum of the integrals I_r is infinite when a tends to zero. Therefore the two magnetostatic force laws differ as expected for two coaxial volume-current distributions.

3. Newton's third law

The main difference between the two force laws is due to the fact that Ampère's law obeys Newton's third law while the Lorentz law does not, as can be verified by comparing equations (3)–(5). This point is indeed fundamental as we will see.

Newton's third law applies to particles in interaction through internal forces in a closed system. In that case only the equality of action and reaction stands. For this closed system we have symmetry and consequently the origin of the frame to describe the interaction can be either one of the two particles or the centre of mass as shown in figure 2. Now, if we introduce a third particle and use the law of superposition for forces, then Newton's third law does not hold anymore since we have applied an external force to our interacting particles, as shown in figure 3. In that case the system consisting of the two interacting particles and the third particle is considered to be an open system. We can distinguish external forces from internal ones by looking at the position of the centre of mass, which is fixed in the laboratory frame for internal forces. This is a general result valid in both classical and relativistic mechanics.

Therefore it is not surprising to have two force laws: the Ampère law for internal forces and the Lorentz law for external forces. In fact we use the Lorentz law for

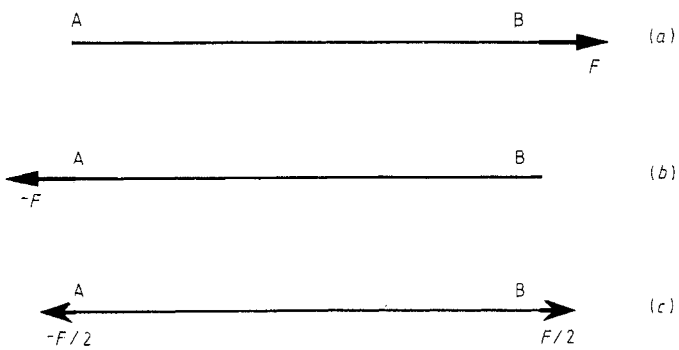

Figure 2. Three possible configurations of the forces for two interacting particles: (*a*) particle A at rest, (*b*) particle B at rest, (*c*) particles A and B in motion.

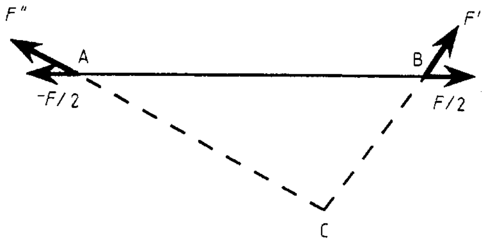

Figure 3. Two interacting particles plus a third particle.

external forces applied to free electrons in vacuum. The distinction between internal and external forces is fundamental and appears in all physics. For example, it is at the heart of the controversy about the outcome of the twin paradox as we have demonstrated [16]. It is also the reason why there are two kinds of radiation, namely spontaneous and stimulated radiation.

Now the current in a conductor is due to the motion of free electrons. Therefore one can apply the Lorentz force to the electrons. However, this approach raises a number of fundamental questions related to this problem.

The first question concerns the neutrality of the conductor which can be at the origin of the resolution of the discrepancy between the two laws as suggested by Whitney [17]. The problem of macroscopic charge neutrality also affects all physics. However, the macroscopic neutrality of particles, such as the neutron, or of atoms, such as the hydrogen atom, and the neutrality of plasma does not imply microscopic neutrality. It is the same as saying that the concept of point charge is not a good concept. For example, we know that the neutron has a magnetic moment. Therefore there is a difference between macroscopic neutrality of a closed system and the absence of microscopic neutrality related to open systems. The question of neutrality of a current-carrying conductor has been addressed in the literature [18–20]. It has been shown [18] that the current density is not totally uniform across the cross section of the conductor.

The second question is: how do the self and non-self Lorentz forces exerted by the magnetic field acting on the drifting electrons transfer the forces to the wire itself? As demonstrated by several authors [21–24] this problem is strongly related to the first question and whether or not the current-carrying conductor forms a closed system. It has also been shown that the Hall effect plays a determining role in the process. As a consequence of this effect the current density will flow parallel to the axis of the conductor in the steady state because there is a balance between the internal transverse forces inside the conductor. Therefore, as pointed out by Whitney [25], the fact that the sum of either longitudinal or transverse forces is zero in the steady state does not mean that tension or compression does not exist inside a conductor. The issue is, rather, whether we are dealing with internal or external forces.

4. Conclusion

We have shown that, indeed, the Ampère and Lorentz force laws are mathematically different. However, they are both physically correct since they address two different systems: the former for a closed system and the latter for an open system. This fact certainly deserves further research since it can bring, in our opinion, a better understanding of the meaning of relativistic theory.

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